Tests of General Relativity and Alternative Theories of Gravity

[1] Michelle Berwa, [2] Prof. Marc Casals

(African Institute for Mathematical Sciences-Ghana)

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Abstract: **General Relativity (GR), formulated by Albert Einstein in 1915, remains the cornefirstone of modern gravitational physics, providing profound insights into the structure and dynamics of the universe. Its predictions have been extensively tested through a wide range of experimental and observational studies, confirming its consistency and predictive power with remarkable accuracy. Nevertheless, ongoing efforts to test GR continue to be motivated by the search for possible deviations, the need to address unresolved cosmological phenomena, and the pursuit of a more fundamental theory of gravity. This essay presents a comprehensive review of past, current, and proposed future experiments designed to test gravitational theories, with particular emphasis on General Relativity. It discusses both the underlying physical principles and the mathematical frameworks that govern these experiments. In addition, the study examines prominent alternative theories of gravity, with specific focus on the Brans–Dicke theory, highlighting their theoretical foundations and observational implications. The analysis aims to evaluate the robustness of General Relativity while exploring the extent to which alternative models can account for observed phenomena, thereby contributing to the ongoing development of gravitational physics.**

Keywords: **General, Relativity, Theories, Gravity.**

1. INTRODUCTION

Albert Einstein's theory of general relativity (GR) was the beginning of the revolution of how scientists perceived the existence of gravity. Prior to this, scientists studied gravity as the force that exists between two massive bodies in space. On the other hand, Einstein described the existence of gravity as the curvature of space-time. This theory of gravity was accompanied by some predictions.

GR predicted the advance perihelion of Mercury's orbit. In 1845, Le Verrier discovered that there is something peculiar with Mercury after studying the motion of all the existing planets. The theory of Einstein was able to test this anomalous and obtained a discrepancy of 43 arc seconds per century in 1967. GR accounted for the anomalous shift naturally without disturbing the agreement with other planetary observations [20].

Again GR predicted the de ection of light by the sun by 1.75 arc seconds for a ray that passes by the sun. In 1919, two different expeditions set out from England; one for Sobral, in Brazil and the other for the Island Principe, in the Gulf of Guinea in Africa.The observations were to be made in a total solar eclipse, during which the moon blocks light from the sun and reveal the star field behind it. The results showed by Sir Frank Dyson and Arthur Eddington revealed the angle at which light bends to be 1.98 ± 0.16 for the Sobral expedition and 1.61 ± 0.40 for Principe expedition [20].

Gravitational Redshift too was another prediction of GR. This was confirmed in 1960 by the Pound-Rebka experiment, in which they compared frequency shift in gravitational field [4]. The above tests are classified as classical tests of GR. GR also predicted the existence of Gravitational waves (GWs), the first confirmation of their existence was observed in 2015 when the Laser Interferometer Gravitational-Wave Observatory (LIGO) received the first signal that confirmed the existence of GWs which is a recent test [17]. Today, technology is advanced that there are more planned observatory apart from LIGO, and VIRGO, such as Laser Interferometer Space Antenna (LISA), DECi-hertz Interferometer Gravitational-wave Observatory (DECIGO), KAmioka GRAvitational (KAGRA) in Japan, GEO600 in Germany, which test the gravitational field in the strong regime and so allow us to contrast observational data with the predictions from General Relativity.

So far, GR has been passing many tests that have been applied to it. However, it fails in some regime; such as at the Bing Bang or geometry at the center of a black hole (BH), where GR predicts divergent quantities [29]. For these reasons, many scientists tried to formulate alternative theories of gravity that reproduce the results of General Relativity or seek to resolve unsolved predictions.

The theory of GR has many applications in everyday technology. This includes the Global Positioning System (GPS) which is used everywhere to locate the position. GPS consists of satellites that send electromagnetic signals. These signals arrive at anyone with a GPS receiver or device. The signal contains information on where the satellite is and what time the signal was sent. Using this information, the GPS receiver calculates the distance from the satellite d = vt, with d signal speed, and send it back to the satellite [13].

Another application is in Cosmology1. This is applied by solving the Einstein field equations with certain assumptions. The solutions tell that as time increase, the size of the universe also increases. As it was observed by Edwin Hubble in 1929; galaxies were recessing away at velocities proportional to their distance from our galaxy $v = H_0 r$, where v is the recession velocity of the galaxy, r is the distance from ours, and H0 is the Hubble parameter that varies in time [23].

This project is as follows; Chapter 1 is the introduction to the essay. In chapter 2, we give an introduction to General Relativity. In chapter 3, we review experimental tests that verify the predictions made in General Relativity. Chapter 4, reviews the Brans-Dicke theory which is an alternative theory to General Relativity. Chapter 5 gives the conclusion to the project.

[1]Cosmology: is a branch of astronomy that studies the origin and evolution of the universe

2. INTRODUCTION TO GENERAL RELATIVITY

2.1 Newton's Theory

Newton's theory is a theory of gravity that played a big role in science. The laws of motion and gravity have been used in the development of mechanics from Newton to Einstein. Astronomers in particular used them in the calculation of motions of the planets. Newton described gravity as an attractive force between any two objects. This is known as the law of universal gravitation [20].

$$F = -G\frac{Mm}{r^2} \quad \text{..... (2.1.1)}$$

Any two bodies in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This is expressed in (2.1.1) where F is the force of attraction, M and m, the masses of the two objects and r, the distance between the two objects.

We de ne the inertial reference frame as a set of coordinates that can be used to describe positions and velocities of the body without considering external force that can act on that body. From Newton's first law of motion, if such body is in motion, it continues to be in motion with a constant velocity or if at rest, continue to stay at rest unless a net force is acted on it.

The second law of Newton states that, in an inertial frame the vector force is equal to the mass m of a body multiply by the acceleration a of that object,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}, \quad \text{..... (2.1.2)}$$

and the third law states that for every action there is always an equal and opposite reaction

$$\vec{F}_{A\to B} = -\vec{F}_{B\to A},$$

where $F\to A\to B$ is the force on body B exerted by body A and vice-versa for $F\to B\to A$.

Definition 2.1.1. Galilean transform is a relation between coordinates axes of two reference frames. Consider two inertial frames; O frame with coordinates (t, x, y, z) and O^r frame with coordinates (t^r, x^r, y^r, z^r) moving with relative velocity v_i with respect to frame O. The relationship between their coordinates is given by [2]

$$t = t^r,$$

$$x = x^r + v_x t,$$

$$y = y^r + v_y t,$$

$$z = z^r + v_z t.$$

2.2 Special Relativity

The special theory of relativity (SR) is a theory of relativity formulated by Einstein in 1905, which studies the relationship between space and time. SR introduces objects by observers in inertial frames, the object may be in motion that is accelerated with respect to the inertial frames in the absence of gravitational field. [22]

The SR theory in its studies of the relationship between space and time found that the two can be merged to form space-time [2]. This is a four-dimensional manifold (t, x, y, z).

Definition 2.2.1. An event is a point in space-time coordinate (t, x, y, z).

In Minkowski space-time, events have coordinates $x^\mu (\mu = 0, 1, 2, 3)$ whereas in Cartesian co- ordinates we have it as $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and in polar coordinates $(x^0, x^1, x^2, x^3) = (t, r, \vartheta, \varphi)$.

The square of the infinitesimal distance between two events in Cartesian coordinates is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad (2.2.1)$$

and can be written in the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \qquad (2.2.2)$$

where $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. In matrix form

$$\eta_{\mu\nu} = \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

In spherical polar coordinates we have the metric written as

$$ds^2 = -dt^2 + dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2,$$

where $\eta_{00} = -1$, $\eta_{11} = 1$, $\eta_{22} = r^2$, $\eta_{33} = r^2 \sin^2 \vartheta$, and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. In matrix form, we write it as [20]

$$\eta_{\mu\nu} = \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 1 & \sin^2 \theta \end{matrix}$$

The classification of separation in space-time is described as:

- timelike when ds^2 is negative,
- spacelike when ds^2 is positive, and
- lightlike or null when ds^2 is zero [18].

[2] Space-time a 4-dimensional coordinate system used to locate objects or events.

2.2.2 Test of Special Relativity

Michelson-Morley experiment measured the speed of light in different inertial frames to detect the existence of ether [22]. Light is emitted from the source reaching the beam splitter *A* (half-silvered mirror), which divides it into two Half beam of light goes to a fully reactive mirror 1, then reacted to *A*, perpendicular to the motion of the earth. The other half beam of light goes to a fully reactive mirror 2 and bounced back to *A* of length L_2 parallel to the motion of earth with respect to the ether. After meeting again they are sent to interferometer *D* to be detected. The following diagram Figure 2.1 illustrates the Michelson- Morley interferometer.

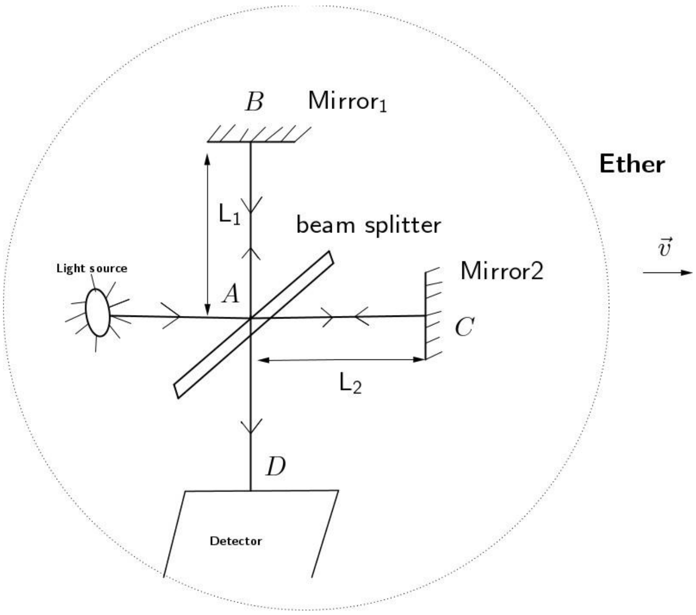

Figure 2.1: Michelson-Morley experiment.

a. Calculation in ether's frame reference:

c is velocity of light beam with respect to ether, *v* is velocity of earth with respect to ether. In path AB, the light wave is propagating in the direction perpendicular to the velocity of the earth through ether, Figure 2.2 shows the shift of the mirror in a vertical motion. The distance *d* is the shifted distance by the mirror as the light beam travels through the medium, shown in Figure 2.2.

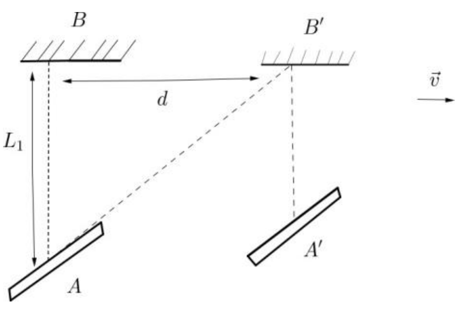

Figure 2.2: View in a vertical motion.

By using Pythagoras theorem, we have

$$(L_{AB'})^2 = L_1^2 + d^2$$

$$(c\, t_{AB'})^2 = L_1^2 + (v\, t_{AB'})^2$$

$$(t_{AB'})^2 (c^2 - v^2) = L_1^2$$

$$t_{AB'} = \frac{L_1}{(c^2 - v^2)^{1/2}}$$

$$t_{AB'} = \frac{L_1}{c(1 - \frac{v^2}{c^2})^{1/2}}.$$

By symmetricity $t_{AB}' = t_B'A'$. Expansion of $\frac{1}{(1-\frac{v^2}{c^2})^{1/2}}$ Gives $1 + \frac{v^2}{2c^2}$, we can ignore high powers of v^2/c^2 because it is too small. Therefore

$$t_{AB'} \approx \frac{L_1}{c}\left(1 + \frac{v^2}{2c^2}\right) \quad (2.2.3)$$

Total time taken by light beam to go to *B* and come back to *A* is

$$t_1 = t_{AB'} + t_{B'A'},$$

$$t_1 \approx \frac{2L_1}{c}\left(1 + \frac{v^2}{2c^2}\right). \quad (2.2.4)$$

For path AC, light is traveling in same direction as the velocity of earth with respect to ether. Distance traveled by light to *AC* is

$$L_{AC} = L_2 + t_{AC}v,$$

$$t_{AC}c = L_2 + t_{AC}v,$$

$$t_{AC}(c - v) = L_2,$$

$$t_{AC} = \frac{L_2}{c - v}.$$

Total time traveled from *A* to *C* and back is

$$t_2 = t_{AC} + t_{CA}$$

$$= \frac{L_2}{c - v} + \frac{L_2}{c + v}$$

$$= \frac{2L_2c}{c^2 - v^2}$$

$$= \frac{2L_2c}{c(1 - \frac{v^2}{c^2})}.$$

Expansion of $\frac{1}{1-\frac{v^2}{c^2}}$, gives $1 + \frac{v^2}{c^2}$ by ignoring high power of v^2/c^2 because it is too small, therefore

$$t_2 \approx \frac{2L_2}{c}\left(1 + \frac{v^2}{c^2}\right) \quad (2.2.5)$$

Time period difference gives

$$\Delta t = t_1 - t_2,$$

$$= \frac{2L_1}{c}\left(1 + \frac{v^2}{2c^2}\right) - \frac{2L_2}{c}\left(1 + \frac{v^2}{c^2}\right)$$

$$= \frac{2}{c}(L_1 - L_2) + \frac{v^2}{c^3}(L_1 - 2L_2). \quad (2.2.6)$$

The time difference found in equation (2.2.6) indicates that t_1 is larger than t_2, therefore there is an interference pattern detected.

b. Calculation in Earth's reference frame: The earth is in the rest frame, light and ether are moving with respect to earth. We have the following new diagram:

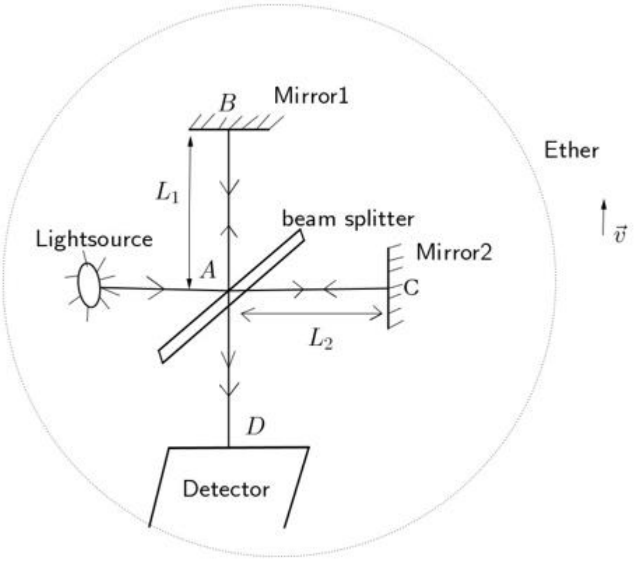

Figure 2.3: Diagram in earth's frame.

This time the light beam which is going to mirror 1 is in the parallel direction motion of ether $\vec{v}$ with respect to earth. One way time of light to go to B is

$$t^r_{AB} = \frac{L_1}{c - v}.$$

To the way back is

$$t^r_{BA} = \frac{L_1}{c + v}.$$

Total time taken by light to mirror 1 and back is

$$t^r_1 = t^r_{AB} + t_{BA}{}^r$$
$$= \frac{L_1}{c - v} + \frac{L_1}{c + v}.$$

Using the expansion we did in the previous frame, we will have the total time taken by light to travel to mirror 1 and back to be

$$t^r_1 \approx \frac{2L_1}{c}\left(1 + \frac{v^2}{c^2}\right) \quad (2.2.7)$$

Light which is traveling to mirror 2 is perpendicular to the motion of ether with respect to earth.

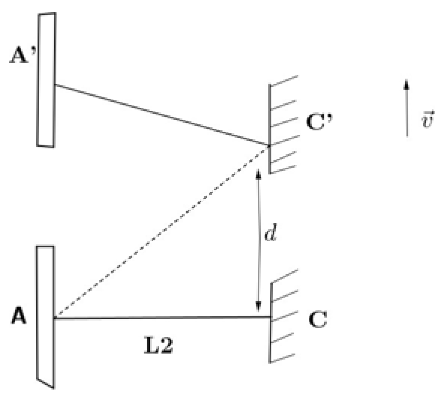

Figure 2.4: View in perpendicular motion.

$$L_{AC'}^2 = L_2^2 + d^2,$$
$$t_{AC'}^2 \cdot c^2 = L_2^2 + t_{AC'}^2 \cdot v^2,$$
$$t_{AC'}^2(c^2 - v^2) = L_2^2$$
$$t_{AC'} = \frac{L_2}{c\left(1 - \frac{v^2}{c^2}\right)^{1/2}}.$$

expansion gives

$$t_{AC'} \approx \frac{L_2}{c}\left(1 + \frac{v^2}{2c^2}\right).$$

By symmetry $t_{AC}' = t_A'C'$ and total time it takes light to travel to mirror 2 is

$$t_2^r \approx \frac{2L_2}{c}\left(1 + \frac{v^2}{2c^2}\right),$$
$$\Delta t^r = t^r_1 - t^r_2,$$
$$= \frac{2L_1}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2L_2}{c}\left(1 + \frac{v^2}{2c^2}\right)$$
$$\Delta t^r = \frac{2}{c}(L_1 - L_2) + \frac{v^2}{c^3}(2L_1 - L_2).$$

This time difference too gave an interference pattern.

The time difference in the two frames is

$$\Delta T = \Delta t - \Delta t^r,$$
$$= \frac{2}{c}(L_1 - 2L_2) + \frac{v^2}{c^3}(2L_1 - L_2) - \frac{2}{c}(L_1 - L_2) - \frac{v^2}{c^3}(L_1 - 2L_2),$$
$$= \frac{2L_1v^2}{c^3} - \frac{L_2v^2}{c^3} - \frac{L_1v^2}{c^3} + \frac{2L_2v^2}{c^3},$$
$$= \frac{L_1v^2}{c^3} + \frac{L_2v^2}{c^3},$$
$$\Delta T = \frac{v^2}{c^3}(L_1 + L_2). \quad (2.2.9)$$

The time difference in ether's frame and earth's frame resulted in equation (2.2.9) means there is a shift, otherwise, the interference pattern recorded should have been the same. Let n be the number of shifted fringes in the pattern, it can be calculated as

$$n = \frac{\Delta T}{T}$$
$$c = \frac{\lambda}{T} \Longrightarrow T = \frac{\lambda}{c}$$
$$n = \frac{\Delta Tc}{\lambda}.$$

Where T is the period of one vibration and λ wavelength, replacing the value of ΔT , we get the wave fringe distance ratio as

$$n = \frac{v^2}{c^2\lambda}(L_1 + L_2).$$

Considering Sodium light source, $\lambda = 5.9 \times 10^{-5}cm$, arm lengths $(l_1 + l_2) = 11m + 11m = 2.2 \times 10^3cm$, earth's velocity through ether $v = 3 \times 10^4 m/s$, Light speed $c = 3 \times 10^8 m/s$

$\frac{v^2}{c^2} = \left(\frac{3\times10^4}{3\times10^8}\right)^2 = 1 \times 10^{-8} m/s$. Then wave fringe distance ratio gives

$$n = \frac{1 \times 10^{-8}}{5.9 \times 10^{-5}}(2.2 \times 10^{3}) \approx 0.37.$$

The conclusion drawn after this experiment is:

1. There was no shift in the interference pattern, because they calculated almost null fringe shifted distance.
2. Light wave have the same velocity as it travels in different frames .
3. Ether does not exist and light does not need a medium to propagate.

2.2.3 Principle of Special Relativity

Newton's law and the square of the distance between two events ds^2 were invariant under Galilean transformation but this resulted in two problems. First, Maxwell's equations were not invariant under Galilean transformation but under Lorentz's transformation they were invariant. Secondly, the speed of light is not invariant under Galilean transformation. The opposite was shown by the Michelson-Morley experiment. In this experiment, the speed of light in different inertial frames as it is moving through a medium was the same, it

was supposed the medium to be ether but it was shown that the medium does not have effect on the speed of light. The value was c which is equal to $3 \times 10^8\ ms^{-1}$. Therefore, Einstein proposed the following postulates [23]:

1. The laws of physics are invariant in all inertial frames[3].
2. The speed of light in a vacuum is constant in all frames of references[4].

2.3 Lorentz Transformations

An event which have space-time coordinates (t, x, y, z) in reference frame system S and (t^r, x^r, y^r, z^r)

coordinates in S^r reference frame system moving at a velocity v with respect to S^r. The relation- ship of these coordinates according to Lorentz transformation is given as [2]

$$t^r = \gamma(t - vx), \quad (2.3.1)$$

$$x^r = \gamma(x - vt), \quad (2.3.2)$$

$$y^r = y, \quad (2.3.3)$$

$$z^r = z, \quad (2.3.4)$$

where γ is the Lorentz factor $\frac{1}{(1-\frac{v^2}{c^2})^{\frac{1}{2}}}$.

By solving for (t, x, y, z) in terms of (t^r, x^r, y^r, z^r), we can easily derive the inverse Lorentz trans- formation as

$$t = \gamma(t^r + vx^r), \quad (2.3.5)$$

$$x = \gamma(x^r + vt^r), \quad (2.3.6)$$

$$y = y^r, \quad (2.3.7)$$

$$z = z^r. \quad (2.3.8)$$

[3]Inertial reference frame: is a frame where there is no external force is acting on a body (non-acceleration reference frame).
[4]Reference frame: is a set of coordinates that can be used to describe positions and velocities of objects in that frame.

The consequences of the theory of special relativity are:

2.3.1 Time dilation

From (2.3.5) a xed clock in O^r frame record two events A and B that occur at the same place but in different times it means $x^r_A = x^r_B$, we have

$$t_A = \gamma(t^r_A + vx^r_A),$$
$$t_B = \gamma(t^r_B + vx^r_A).$$

The time interval with respect to frame O is

$$\begin{aligned} \Delta t &= t_B - t_A, \\ &= \gamma(t^r{}_B - t^r_A), \\ \Delta t &= \gamma \Delta t^r. \end{aligned} \qquad (2.3.9)$$

Equation (2.3.9) tells that the time interval Δt measured by an observer moving with respect to the clock is longer than the time interval Δt^r measured by the observed at rest with respect to the clock because the Lorentz factor $\gamma > 1$.

2.3.2 Length contraction

From (2.3.2), if two events occur simultaneously[5] at a reference frame O^r but in different places, it means $t_A = t_B$. This can be written as

$$\begin{aligned} x^r_A &= \gamma(x_A - vt_A), \\ x^r_B &= \gamma(x_B - vt_A). \end{aligned}$$

The length difference according to O reference frame is

$$\begin{aligned} \Delta x^r &= x^r{}_B - x^r_A, \\ &= \gamma(x_B - x_A), \\ &= \gamma \Delta x, \\ \Delta x &= \frac{\Delta x^r}{\gamma}. \end{aligned} \qquad (2.3.10)$$

Equation (2.3.10) shows that the length of a body in motion is reduced by a factor γ.

2.4 Equivalence of Masses

Normally, mass is described as the amount of matter in an object. In Newtonian theory, we can ascribe three masses to anybody which describes different properties of the body. These properties include the following:

2.4.1 Inertial mass m^I

This is a measure of a body's resistance to change of motion. Newton's second law in (2.1.2), the inertial is

$$m^I = \frac{F}{a}, \qquad (2.4.1)$$

the higher the inertial mass, the harder to change the object's state of motion.

[5]Simultaneously: the two events occur at the same time (i.e $tA = tB$).

2.4.2 Passive gravitational mass m^P

This is also a measure of a body's reaction to a gravitational field. Let us consider gravitation potential φ at some point. Then m^P is placed at this point. The force that it will experience is given by

$$F = -m^P \nabla \varphi. \qquad (2.4.2)$$

2.4.3 Active gravitational mass m^A

This is a measure of the strength of the gravitational led produced by a body. If m^A is placed at origin, then the gravitational potential at any distance r is given by

$$\varphi = -\frac{Gm^A}{r}. \qquad (2.4.3)$$

According to Newtonian framework, these three masses relate each other.

To relate inertial mass and passive gravitational mass, let us assume two particles of inertial masses m_1^I and m_2^I and passive gravitational masses m_1^P and m_2^P are dropped from the same height in the gravitational field. Then, from (2.4.1) and (2.4.2), we have

$$m_1^I a_1 = -m_1^P \nabla\varphi,$$
$$m_2^I a_2 = -m_2^P \nabla\varphi.$$

From the two equations we can nd $\nabla\varphi$ as

$$-\nabla\varphi = \frac{m_1^I a_1}{m_1^P},$$
$$-\nabla\varphi = \frac{m_2^I a_2}{m_2^P}.$$

Equating these values we see that

$$\frac{m_1^I a_1}{m_1^P} = \frac{m_2^I a_2}{m_2^P}.$$

It is worth noting that a1 = a2 because there is no air resistance they falls with the same acceleration. Then we get

$$m_1^I/m_1^P = m_2^I/m_2^P.$$

Equation (2.4.4) says that the ratio $\frac{m^I}{m^P}$ is equal to a universal constant, without loss of generality.

Taking that constant to be 1 we obtain this result: "Inertial mass is equal to passive gravitational mass."

In order to relate active gravitational mass and passive gravitational mass, let us consider two isolated bodies situated at points *Q* and *R*. From (2.4.3), the gravitational potential due to each other is

$$\varphi_1 = -\frac{Gm_1^A}{r}, \quad \text{and } \varphi_2 = -\frac{Gm_2^A}{r}.$$

From (2.4.2) the force which each body experiences is

$$F_1 = -m_1^P \nabla\varphi_2, \quad \text{and } F_2 = -m_2^P \nabla\varphi_1.$$

If we take *Q* to be the origin, the gradient operators are

$$\nabla_R = \hat{r}\frac{\partial}{\partial r} = -\nabla_Q,$$

$$F_1 = \frac{Gm_1^P m_2^A}{r^2}\hat{r}, \text{ and } F_2 = -\frac{Gm_2^P m_1^A}{r^2}\hat{r}.$$

By Newton's third law, *F*1 = —*F*2, therefore

$$m_1^P/m_1^A = m_2^P/m_2^A. \tag{2.4.5}$$

Equation (2.4.5) says that the ratio $\frac{m^P}{m^A}$ is equal to a constant. This implies that the active gravitational mass is equal to passive gravitational mass.

Galilean experiment says that if we neglect non-fundamental forces like air resistance, two bodies dropped from the same height will reach the ground together irrespective their internal composition. In other words they su er the same acceleration. This was proved by Dave Scott (an astronaut) who tried this experiment on the moon where there is no atmosphere. He confirmed that a hammer and feather reach the ground at the same time. This is why in Newtonian theory we simply refer to the mass m of a body, where

$$m = m^I = m^A = m^P$$

This relationship is resulting from transitivity between equations (2.4.4) and (2.4.5).

2.5 General Relativity

Einstein formulated a general theory of relativity in 1915. SR was only used to place where the space-time is at (no gravitational field). He realized that space-time curves in the presence of massive object or energy [4].

From (2.2.2), we have seen that the line element ds^2 of the space-time in terms of the metric tensor $\eta_{\mu\nu}$ is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.5.1)$$

where dx is a infinitesimal displacement in the μ and ν's direction, and runs in range $\mu, \nu = 0, 1, 2, 3$ because of the four-dimension of space and time.

The relation between this curvature and the distribution of massive object present is described by the the Einstein's field equations of general relativity [20]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2.5.2)$$

where $R\mu\nu$ is the Ricci tensor, R is the Ricci curvature scalar, $g\mu\nu$ is the metric tensor,

G is the Newton's gravitational constant,

c is the speed of light, and

$T\mu\nu$ is the energy-momentum tensor.

For small deviations of nonzero curvature

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O([h_{\mu\nu}]^2).$$

where $\eta\mu\nu = \text{diag}(-1, 1, 1, 1)$ is a space-time metric tensor and $h\mu\nu$ is the perturbation metric tensor, and $h\mu\nu$ 1. This implies that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (2.5.3)$$

The left hand side of Einstein's equation (2.5.2) represents the geometry of space-time expressed in terms of the metric and Riemann tensor

$$R_{\mu\nu} = R^{\beta}_{\mu\beta\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \quad (2.5.4)$$

We define a at manifold as one in which $R^{\circ}_{\mu\beta\nu} = 0$, otherwise is curved.

In general relativity, material objects acted on by gravity follow paths in space-time described by the geodesic equation [2]

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad (2.5.5)$$

where $\Gamma^{\mu}_{\alpha\beta}$ are Christo el symbols defined as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} (g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) \tag{2.5.6}$$

Matter tells space-time how to curve, and space-time tells matter how to move - John Wheeler. Einstein's equation (2.5.2) gives ten equations in in four-dimensional space-time because of symmetricity some are equal. The labels for $\mu\nu$ take values 0, 1, 2, 3. The number of equations is given by

$$\frac{n(n+1)}{n}$$

where n is 4.

2.5.1 Principle of general relativity

GR includes gravitational field and studies motion of bodies in curved space-time. It is not easy to distinguish between accelerated reference frames and gravitational field [20].

Consider two reference frames; inertial reference frame O (i.e non-accelerated frame) in which there is gravitational led and the second reference frame O^r which is accelerated uniformly with respect to inertial frame but without a gravitational field. The principle of equivalence proposed by Einstein says that The results of experiments in the two frames should be the same [18].

For example, consider a lift with a person and a massive object inside.

Case 1: In space, ignoring external forces acting on the lift. The lift is placed in a spaceship which is accelerated with a constant acceleration g relative to an inertial observer and the person releases the object, it will fall on the ground of the lift with acceleration g.

Case 2: The same lift is placed on the surface of the earth, ignore rotational and orbital motions. The object released falls to the ground with acceleration g.

The weak equivalence principle state that a trajectory of a freely falling body is independent of its internal structure and composition. It means all bodies fall with the same acceleration regardless of their composition in gravitational field [2].

1. The Einstein equivalence principle (EEP) in short states that:
   a. The WEP is valid for tests bodies[6].
   b. For any locally non-gravitational experiment, i.e. experiments performed inside the laborator and no interactions with external fields other than gravitational field:
      i. the outcome is independent of the velocity of the freely falling reference frame (local Lorentz invariance in which it is performed (LLI)).
      ii. The outcome is independent of position and time in the universe it is performed (local position invariance (LPI)).
2. The strong equivalence principle states that for both test and self-gravitational bodies[7]:
   a. The WEP is valid.
   b. The outcome of any local test experiment is independent of the velocity of the apparatus.
   c. The outcome of any local experiment is independent of where and when in the universe it is performed [1].\

[6]Test bodies are bodies that are not acted upon by such forces as electromagnetism and too small to be affected by tidal gravitational forces.

[7]Self gravitating bodies are massive bodies such as planets and stars.

3. TESTS OF GENERAL RELATIVITY

Einstein's theory of general relativity so far has passed all experimental texts applied to it. However, different scientists keep on to test the more with hope to discover a new theory [4]. In this chapter, we will review three classical tests of GR; gravitational Redshift, detection of light, the advance perihelion of Mercury's orbit, and a current test of detecting gravitational waves.

3.1 The Gravitational Redshift

Gravitational Redshift in Einstein's general theory of relativity is a phenomenon by which clocks run at different rates in different places of a gravitational fifield. Redshift is any decrease in the frequency of photon rays propagating towards a region of lower gravitational potential. Blueshift on the other hand is when rays of a photon propagate towards a region of higher gravitational potential and this implies an increase in frequency [27].

The Gravitational Redshift experiment tests the principle of local position invariant (LPI). The LPI states that the outcome of any non-gravitational local experiment is independent of when and where in the universe it is performed [18]. The experiment measures the frequency or wavelength shift between two identical clocks placed at different heights in a static gravitational fifield. Redshift is also known as Doppler effect [8].

The fractional frequency difference between two identical clocks placed at different places is

$$z = \frac{\Delta v}{v}$$

To analyze what effect gravity has on the photon, let us consider a photon emitted by an atom *A* at rest in a non-accelerating frame *S* in which there is gravitational field. A photon freely falls down a distance *d* with a uniform gravitational field *g* directed downward in *S* to reach the detector *D* at rest, as shown in Figure 3.1(a). The frame *Sr* as shown in 3.1(b), there is no gravitational led *Sr* is accelerating upward relative to *S* frame with acceleration *a*. Atom *A* and detector *D* are moving towards each other with constant velocity in frame *Sr*.

The following diagram shows the gravitational effects of gravity on a photon [22].

[8]Doppler effect: A phenomenon where light waves produced by a star towards an observer, it compressed together, giving a high-frequency light waves and when it is moving away to the observer, it stretched out giving a low frequency.

On the electromagnetic spectrum, blue color represents higher-frequency visible light, short wavelength, and the red color represents low frequency.

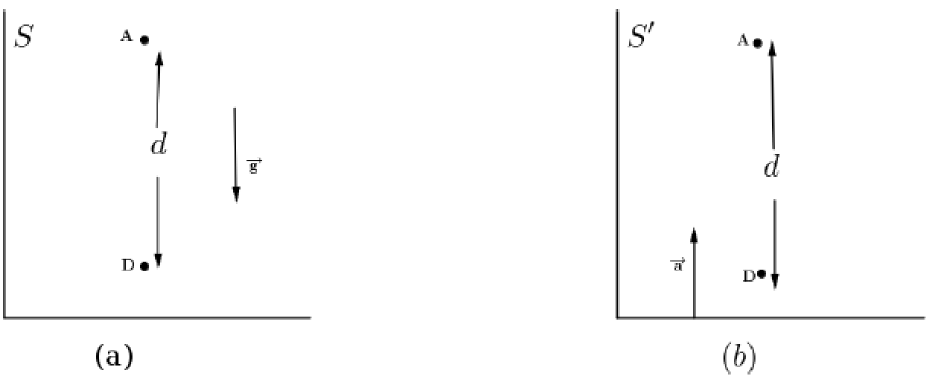

Figure 3.1: Photon in two different frames

In frame *Sr*, atom *A* is moving with speed *u* towards detector *D* where *t* is the time of ight of the photon ($t = d/c$). The atom emits a photon of speed *c*. The detector *D* has an approach velocity relative to the atom of $v = gd/c$, independent of *u*. Hence, the frequency received, *vr*, is greater than that emitted, *v*. The Doppler e ect in Newtonian theory when two objects move toward one another gives the following formula

$$\frac{v^r}{v} = \sqrt{\frac{c+v}{c-v}}.$$

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Substituting the value of v we get

$$\frac{v^r}{v} = \sqrt{\frac{c + gd/c}{c - gd/c}} = \sqrt{\frac{c(1 + gd/c^2)}{c(1 - gd/c^2)}}.$$

Multiply the factor that gives 1, we get

$$\frac{v^r}{v} = \sqrt{\frac{1 + gh/c^2}{(1 - gd/c^2)(1 + gd/c^2)}} = \sqrt{\frac{1 + gd/c^2}{1 - (gd/c^2)^2}}$$

$$\approx 1 + gd/c2$$

$$z = \frac{\Delta v}{v} = \frac{v^r - v}{v} = \frac{gd}{c^2}. \tag{3.1.1}$$

Equation (3.1.1) shows the frequency difference in *Sr* frame. Where v is the emitted frequency, vr is the received frequency, g is acceleration of gravity, d is the distance between emitter and receiver, and c is the speed of light.

By the principle of equivalence, the results obtained in frame *Sr* should be the same for *S* frame. In *S* frame a photon of energy $E = kv$ freely falls from A to D in a gravitational field. From Einstein energy mass equivalence relation that mass is given by $E = mc^2$. We can associate gravitational mass to a photon from the two energies, that is

$$kv = mc^2,$$
$$m = \frac{kv}{c^2},$$

where m is the associated gravitational mass to a photon that has energy, k is the Plank's constant, and c is the speed of light. The falling photon gains energy of mgd, where m is the mass ascribed on the photon, g is the acceleration due to gravity. It reaches to detector D with energy $E^r = kv^r$

By conservation of energy, total emitted energy is equal to the total energy received. Therefore

$$kv^r = kv + mgd,$$
$$kv^r = kv + \frac{kv}{c^2} gd$$
$$v^r = v + \frac{vgd}{c^2}$$
$$\frac{v^r}{v} = 1 + \frac{gd}{c^2}$$
$$\frac{\Delta v}{v} = \frac{gd}{c^2}. \tag{3.1.2}$$

The results obtained in *S* frame (3.1.2) is the same as that obtained in S^r frame (3.1.1) [22].

In 1960, Robert Pound and Glen Rebka in their experiment shot gamma-rays of radioactive iron up the side of a tower of 74*ft* which is equivalent to 22.5m high at Havard University. They measured the fractional frequency different $\frac{\Delta v}{v}$ between two identical clocks placed at different places in a static gravitational field [2].

From equation (3.1.1),

$$\frac{\Delta v}{v} = \frac{9.8m/s^2 \times 22.5m}{(3 \times 10^8 m/s)^2} \approx 2.5 \times 10^{-15}.$$

In their observations, it was confirmed that the results of frequency depend on the gravitational potential between emitter and receiver.

3.2 Perihelion Advance of Mercury

In 1859, Le Verrier noticed that the elliptical orbit of Mercury is unusual when he was doing his studies on the motion of planets. Perihelion is a point of a planet's orbit closest to the sun while the point that is far from the sun is called aphelion. Mercury is the closest planet to the sun, and the eccentricity[9] of the orbit for Mercury and Pluto are the highest compared to other planets. This makes their perihelion to be distinguished from the others.

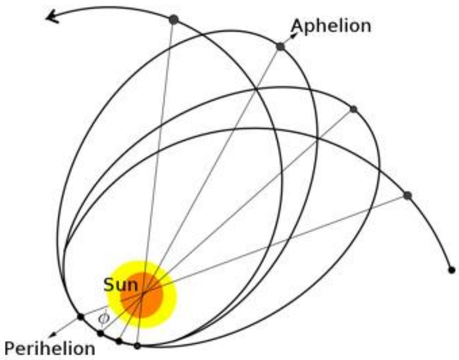

Figure 3.2: Precession of Mercury's orbit

The motion of planets and of light in gravitational fifield of the sun in Schwarzchild space-time is given by geodesic equation. We have seen that the geodesic equation (2.5.5) in Chapter 2. Newton's theory of gravity predicts the orbit of Mercury around the sun to be perfect ellipse if it were isolated from other planets. The differential equation for the orbit in general relativity is given by

$$\frac{d^2}{d\varphi^2}\frac{1}{r} + \frac{1}{r} = \frac{mc^2}{a^2} + \frac{3m}{r^2}, \quad (3.2.1)$$

where φ is the inclination of the orbit of planets with respect to the sun, r is the radial parameter, a is the semi-major axis, and constant $m = \frac{GM}{c^2}$, where M is the mass of the sun, G is gravitational constant, and c speed of light [20].

The differential equation for a planet moving non-circular orbit in Newtonian theory is given by

$$\frac{d^2}{d\varphi^2}\frac{1}{r} + \frac{1}{r} = \frac{mc^2}{a^2} \quad (3.2.2)$$

taking $u = 1/r$, $p = a2/GM = a2/mc2$, p is the semi-latus rectum. The ODE becomes

$$\frac{d^2u}{d\varphi^2} + u = \frac{1}{p} \quad (3.2.3)$$

Equation (3.2.3) is a non homogeneous second order ODE. Solving the homogeneous equation, that is

$$u'' + u = 0,$$

[9]Eccentricity: Parameter that determines the amount by which the orbit of a planet deviates from a perfect circle.

where u' is the first derivative and u'' is second derivative. Let

$$u = \exp(z\varphi)$$
$$u' = z\exp(z\varphi)$$
$$u'' = z^2\exp(z\varphi)$$
$$z^2\exp(z\varphi) + \exp(z\varphi) = 0$$
$$z^2 + 1 = 0$$
$$z = \pm i.$$

We have complex roots. Therefore, the complementary solution (*uc*) of equation (3.2.4) is

$$u_c(\varphi) = A\cos\varphi + B\sin\varphi.$$

Applying the initial conditions, let the first derivative be equal to *e/p* and second derivative be equal to zero. Then

$$u(0) = e/p \implies A = e/p$$
$$u'(0) = 0 \implies B = 0.$$

The complementary solution to the equation then becomes

$$u_c = \frac{e}{p}\cos\varphi. \tag{3.2.5}$$

Since the right hand side of (3.2.3) is a constant, the first and second derivative gives zero. Therefore the particular solution is

$$u_p = \frac{1}{p}.$$

Then, the general solution of equation (3.2.3) is equal to complementary solution plus particular solution, that is

$$u = \frac{1}{r} = \frac{1}{p}(1 + e\cos\varphi) \tag{3.2.6}$$

Equation (3.2.6) is the equation of ellipse in Newton theory, with eccentricity *e*. If *e* = 0, the ellipse becomes a perfect circle.

The GR equation for the orbit in (3.2.1) is composed with the Newtonian equation (3.2.2) and an additional term. Substituting the particular solution (3.2.6) into this term of equation (3.2.1) we get

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{mc^2}{a^2} + 3m\left(\frac{1}{p}(1 + e\cos\varphi)\right)^2$$

Neglecting term in e^2 we have

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{mc^2}{a^2} + 3m\frac{1}{p^2}(1 + 2e\cos\varphi)$$

However replacing the value of $p = a2/mc2$ gives

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{mc^2}{a^2} + \frac{3m^3c^4}{a^4}(1 + 2e\cos\varphi)$$
$$\approx \frac{mc^2}{a^2} + \frac{6m^3c^4}{a^4}e\cos\varphi. \tag{3.2.7}$$

The solution to this new equation (3.2.7) is; the solution of

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{mc^2}{a^2},$$

which obtained before in (3.2.6), plus the solution for the additional term

$$\frac{d^2}{d\varphi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{6m^3c^4}{a^4}e\cos\varphi. \tag{3.2.8}$$

To find the solution of (3.2.8), we guess

$$u(\varphi)=A\varphi\cos\varphi+B\varphi\sin\varphi, \tag{3.2.9}$$
$$u'(\varphi)=A\cos\varphi-A\varphi\sin\varphi+B\sin\varphi+B\varphi\cos\varphi, \tag{3.2.10}$$
$$u''(\varphi)=A\sin\varphi-A\sin\varphi-A\varphi\cos\varphi+B\cos\varphi+B\cos\varphi-B\varphi\sin\varphi. \tag{3.2.11}$$

Then, $u''+u=\frac{6m^3c^4}{a^4}e\cos\varphi$

Equating the factors, we get $A = 0$ and $B=\frac{3m^3c^4}{a^4}e.$ therefore

$$u(\varphi)=\frac{3m^3c^4}{a^4}e\varphi\sin\varphi.$$

So the solution to (3.2.7) is

$$\frac{1}{r}=\frac{mc^2}{a^2}(1+e\cos\varphi)+\frac{3em^3c^4}{a^4}\varphi\sin\varphi \tag{3.2.12}$$

Which can be reduced as follows

$$\frac{1}{r}=\frac{mc^2}{a^2}\left(1+e\cos\left(\varphi\left(1-\frac{3m^2c^2}{a^2}\right)\right)\right)+O\left(\frac{m}{r}\right)^2$$

The perihelion occurs when 1/r is a maximum, that is when

$$\cos\left(\varphi\left(1-\frac{3m^2c^2}{a^2}\right)\right)=1.$$

Hence, φ = 2nπ, with n = 0, 1, 2, . . .

$$\varphi=0,\frac{2\pi}{1-\frac{3m^2c^2}{a^2}},\frac{4\pi}{1-\frac{3m^2c^2}{a^2}},\ldots$$

From Taylor expansion, $(1-x)^{-1}=1+x+x^2+\ldots$

Therefore,

$$\begin{aligned}\varphi &\approx 0, 2\pi\left(1+\frac{3m^2c^2}{a^2}\right)\\ &\approx 0, 2\pi+\frac{6\pi m^2c^2}{a^2}\\ &\approx 0, 2\pi+\delta\varphi,\ldots,\end{aligned}$$

where $\delta\varphi$ is the precession of the ellipse.

Einstein's formula for precession of the ellipse $\delta\varphi$ in one revolution is

$$\delta\varphi=\frac{6\pi m^2c^2}{a^2}$$

Substituting $p = a^2/mc^2 = a2/GM$, where the quantity p represents the semi-latus rectum of the ellipse. It is also given by $p = a_0(1-e^2)$, where a_0 is the semi major axis and e the eccentricity of the orbit. Therefore the precession becomes

$$\delta\varphi = \frac{6\pi m}{p} = \frac{6\pi MG}{c^2 a_0(1-e^2)} \quad (3.2.13)$$

where M is the solar mass ($M = 2 \times 10^{30} kg$) , G is the universal gravitational constant ($G = 6.67 \times 10^{-11} m3/kg\ s$), c speed of light ($c = 3 \times 10^8 m/s$), a_0 is the semi major axis ($a_0 \approx 5.8 \times 10^{10} m$) and e the eccentricity of orbit ($e = 0.21$).

Substituting the values of these constants in equation (3.2.13), gives

$$\delta\varphi = \frac{6 \times \pi \times 6.67 \times 10^{-11} m^3/kg\ s \times 2 \times 10^{30} kg}{(3 \times 10^8 m/s)^2 \times 5.8 \times 10^{10} m\ (1-(0.21)^2)}$$
$$= 1.6040729 \times 10^{-7}\pi \text{ radians}$$
$$\approx 0.1''.$$

The value of the precession in one revolution is 0.1039440216 arc seconds. Since Mercury orbits around the sun in 88 days, in one hundred Earth years it will be

$$\frac{0.1039440216'' \times 365 \text{ days/ year} \times 100 \text{ years}}{88 \text{ days}} \approx 43.1''$$

The solution to (3.2.14) shows the precession of Mercury to be approximately 43.1 arc seconds per century predicted by Einstein theory of general relativity [2].

3.3 Detection of Light

Einstein's theory of general relativity predicted that. The gravitational fifield of the Sun can detect light rays coming from a background star on its way. This is illustrated in Figure 3.3.

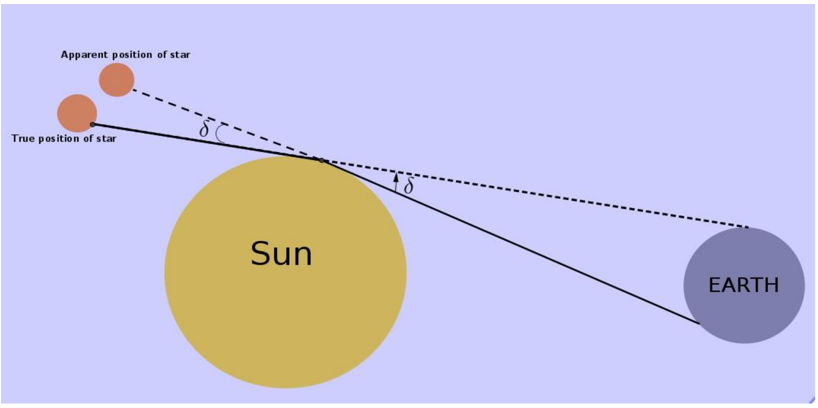

Figure 3.3: Detection of light

To verify this, a total solar eclipse[10] was needed such that the stars near the Sun's rim could be seen as well as the way they move [2]. The theory predicted a detection angle δ of 1.75 arc seconds for the shift of star's position.

In general relativity when light is considered, we can calculate the detection angle by using equation

$$\frac{d^2}{d\varphi^2}\frac{1}{r} + \frac{1}{r} = \frac{3m}{r^2} \quad (3.3.1)$$

Equation (3.3.1) is a non linear second order differential equation, where φ is the inclination of the orbit, and $m = GM/c2$, where M is the mass of the sun, G is gravitational constant, and r ir radial parameter. Since the right hand side is different from zero it is non homogeneous.

To solve it let us first consider the homogeneous equation, that is the right hand side to be zero

$$\frac{d^2}{d\varphi^2}\frac{1}{r} + \frac{1}{r} = 0. \qquad (3.3.2)$$

Taking u = 1/r the solution of equation (3.3.2) is

$$u(\varphi) = A\cos\varphi + B\sin\varphi$$

as we have seen earlier for the solution of equation (3.2.4).

Applying the following initial conditions; that is, the fifirst derivative be equal to $1/r_0$, and the second derivative be equal to zero, where r_0 is the distance from the sun at which light passes, we have

$$u(0) = 1/r_0 \implies A = \frac{1}{r_0},$$
$$u' = -A\sin\varphi + B\cos\varphi = 0,$$
$$u'(0) = 0 \implies B = 0,$$

[10]Solar eclipse: A natural phenomenon occurring when the Sun, Moon, and Earth are aligned.

we get the complementary solution to be

$$u_c = \frac{1}{r} = \frac{1}{r_0}\cos\varphi. \qquad (3.3.3)$$

Substituting equation (3.3.3) in the right hand side of equation (3.3.1), we have

$$\frac{d^2}{d\varphi^2}\frac{1}{r} + \frac{1}{r} = \frac{3m}{r_0^2}\cos^2\varphi.$$

Which can be written as

$$u'' + u = \frac{3m}{r_0^2}\frac{1+\cos 2\varphi}{2}. \qquad (3.3.4)$$

To find the particular solution that is the solution for the right hand side, we guess

$$u = A + B\cos 2\varphi + C\sin 2\varphi, \qquad (3.3.5)$$

where *A*, *B*, and *C* are constants. The first and second derivative of equation (3.3.5) gives

$$u' = -2B\sin 2\varphi + 2C\cos 2\varphi,$$
$$u'' = -4B\cos 2\varphi - 4C\sin 2\varphi.$$

Therefore equation (3.3.4) becomes

$$-4B\cos 2\varphi - 4C\sin 2\varphi + A + B\cos 2\varphi + C\sin 2\varphi = \frac{3m}{r_0^2}\frac{1+\cos 2\varphi}{2}$$
$$A - 3B\cos 2\varphi - 3C\sin 2\varphi = \frac{3m}{r_0^2}\frac{1+\cos 2\varphi}{2}.$$

Equating factors, we get $A = \frac{3m}{2r_0^2}$, $B = -\frac{m}{2r_0^2}$ and $C = 0$.

Substituting values of A, B and C in (3.3.5), we get

$$u = \frac{3m}{2r_0^2} - \frac{m}{2r_0^2}\cos 2\varphi$$
$$= \frac{2m}{2r_0^2} + \frac{m}{2r_0^2} - \frac{m}{2r_0^2}\cos 2\varphi$$
$$= \frac{m}{r_0^2} + \frac{m}{r_0^2}\frac{1-\cos 2\varphi}{2}$$
$$= \frac{m}{r_0^2} + \frac{m}{r_0^2}\sin^2\varphi.$$

The particular solution is

$$u_p = \frac{m}{r_0^2}\left(1 + \sin^2\varphi\right). \tag{3.3.6}$$

The general solution of equation (3.3.1) is the complementary solution plus the particular solution. That is

$$u = \frac{1}{r_0}\cos\varphi + \frac{m}{r_0^2}\left(1 + \sin^2\varphi\right). \tag{3.3.7}$$

Now consider the asymptotes, it means the property of the detection angle as the radial parameter goes to infinity. For equations (3.3.3): $r \to \infty$, $\varphi \to \pi/2$, and for (3.3.7): $r \to \infty$, $\varphi \to \pi/2 + \delta$ From trigonometric identities

$$\cos(\pi/2 + \delta) = -\sin\delta,$$
$$\sin(\pi/2 + \delta) = \cos\delta,$$

where δ is the angle at which light detects when light ray passes near the sun. Equation (3.3.7) becomes

$$0 = \frac{1}{r_0}\cos(\pi/2 + \delta) + \frac{m}{r_0^2}\left(1 + \sin^2(\pi/2 + \delta)\right) \tag{3.3.8}$$

Therefore,

$$0 = -\frac{1}{r_0}\sin\delta + \frac{m}{r_0^2}(1 + \cos^2\delta)$$
$$0 = -\frac{1}{r_0}\sin\delta + \frac{m}{r_0^2}(1 + 1 - \sin^2\delta)$$
$$0 = -\frac{1}{r_0}\sin\delta + \frac{2m}{r_0^2} - \frac{m}{r_0^2}\sin^2\delta.$$

For small angle, $\sin\delta = \delta$,

$$0 = -\frac{1}{r_0}\delta + \frac{2m}{r_0^2} - \frac{m}{r_0^2}\delta^2.$$

Neglecting the term in δ^2 and multiply r_0^2 both sides, we get the detection angle δ of light ray that passes near the sun to be

$$-r_0\delta + 2m = 0$$
$$\delta = \frac{2m}{r_0}.$$

Recall that $m = GM/c^2$, and r_0 is the distance of nearest approach to the origin becomes the radius of the sun. From Figure 3.3, we have two δ angles, therefore, the detection angle becomes

$$\Delta = 2\delta = \frac{4m}{r_0} = \frac{4GM}{Rc^2} \tag{3.3.9}$$

where G is the gravitational constant ($G = 6.67 \times 10^{-11} m^3/kg\ s$), M is the mass of the Sun ($M = 2 \times 10^{30} kg$), R the radius of the Sun ($R = 6.96 \times 10^8 m$) and c the speed of light (3×10^8m/s).

$$\Delta = \frac{4 \times 6.67 \times 10^{-11} m^3/kg\ s \times 2 \times 10^{30} kg}{6.96 \times 10^8 m \times (3 \times 10^8 m/s)^2}$$
$$= \frac{23}{2700000}\ \text{radians}$$
$$\approx 1.75''.$$

The calculations gave a detection angle of approximately 1.75 arc seconds.

On 29th May 1919, ve minutes total solar eclipse played a big role in science. Sir Frank Dyson and Eddington Arthur with their team were able to observe and take photographs that helped them to study the detection of light. Two different expeditions to observe a solar eclipse were set out from England; one for Sobral, in Brazil, and the other for the Principe

Island, in the Gulf of Guinea in Africa. They reported a detection of the angle at which light bends to be 1.98 ± 0.16" for the Sobral expedition and 1.61 ± 0.40" for Principe expedition. Eddington said that the predictions of GR were compatible with the results of their observed detection angle [20].

After the 1919th eclipse, many expeditions continued to test the detection of light predicted by GR, many were not successful but the eclipse of June 30th, 1973 an organized expedition to the Mauritania desert gave a detection angle of 1.66 ± 0.19 arc seconds [11]. In present times, a team of amateur astronomers leads by Bruns, Berry and Dittrich were prepared for the eclipse of August 21, 2017, they measured a detection angle of 1.7512". They are expecting another eclipse in 2024 where they will be having developed tools for the verification of the GR prediction with high accuracy [15].

3.4 Detection of Gravitational Waves

3.4.1 Gravitational Waves

These are transverse waves[11] that travel at the speed of light. Gravitational waves are the results of disturbances in space-time during massive events such as supermassive black hole mergers or collisions between two black holes[12] that are billion times bigger than the sun. Gravitational waves are the powerful explorer to the discovery of new parts of the universe that are invisible by other means, such as black holes, the Big Bang, and other, as yet unknown, objects [17].

[11]Transverse wave: is a wave whose oscillations are travels perpendicular to the wave direction.

[12]Black holes: A region of space-time which has high gravitational field so that no matter or radiation can escape it.

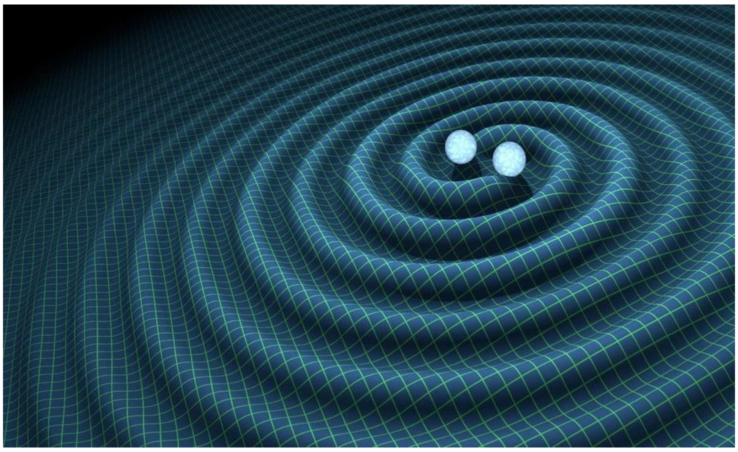

Figure 3.4: Gravitational waves

Einstein in his theory of general relativity predicted the existence of these Gravitational waves. The fifirst confirmation of their existence was observed in 2015 when the Laser[13] Interferometer Gravitational-wave Observatories (LIGO) announced the reception of the fifirst signal. There was a confusion over whether they were real or artifacts of general covariance [8].

3.4.2 LIGO and VIRGO

The Laser Interferometer Gravitational-wave Observatories (LIGO) and VIRGO, are two ground-based observatories that announced the reception of the fifirst signal of gravitational waves. The signal was named after the date on which it was received, *GW* 150914. Where GW simply means Gravitational Waves, then 15 for the year 2015 and 09 for the month September. The number 14^{th} represents the day. It was originated from a binary black hole merger [17].

LIGO and VIRGO laboratories are now among new unique explorers with high technology to help in the study of gravitational waves to discover new parts of the universe. Their goals are to directly observe, detect and measure gravitational waves for scientific researches in hope of knowing better our universe. Also they aim at discovering a new theory of Astronomy with the outlook of the universe than what we know from current telescopes [8]. LIGO has two identical labs of 3000*km* apart; LIGO Hanford, Washington, and LIGO Livingston, Louisiana. This is to ensure that the received signal is really a gravitational wave not a local noise from the earth such as earthquakes and any other acoustic noise. Apart from LIGO in The United States and VIRGO in Cascina, Italy that are ground-based observatories, there are other GW observatories in the world. These include GEO600 in Hannover, Germany, with 600 *m* arms interferometer that makes it low sensitive. There is also an under-ground based observatory in Japan called KAmioka GRAvitational (KAGRA) detector which has 3 *km* arms length interferometer [16].

3.4.3 LIGO's interferometer

LIGO lab is made up of the two interferometers[14], each with two arms in the form of *L*, extending 4 *km* in length. The size is key because of the tiny stretching and squeezing effect that increases in proportional to the link of the interferometer. LIGO's interferometer acts as antennae (ground-based antennae) to detect GWs. They have the capability of detecting and measuring the gravitational waveforms from astronomical sources in a frequency band between 10 Hz and 1000 Hz. The longer they are, the easier the signals are to detect [4].

[13]Laser: Light Amplification by Stimulated Emission of Radiation
[14]Interferometer: A device that works by splitting light into two separate beams that travel different light paths along two arms.

Figure 3.5 shows the pictorial view of the observatories. LIGO is shown by 3.5a with the left image showing the LIGO observatory in Hanford and the right that of Livindston [17]. VIRGO is shown in 3.5b [5].

(a) (b)

Figure 3.5: Pictures of the LIGO and VIRGO observatories

3.4.4 Frequency and working principle

Gravitational waveform is splitted into two by the interferometer, propagated inside arms (tubes) and bounced back by polished mirrors. Unlike the Michelson-Morley's original interferometer where the light beam was only traveling 11 meters, this one the laser travels in the 4 km arms length. The distance of mirrors changes; one arm becomes short while the other one is long and instant later they switch. As the distance between mirrors varies, so does the alignment of the pattern varies to a constructive pattern. Then the detector measures intensity of signal received, thus, measures the GWs [7]. This is illustrated in Figure 3.6 [17].

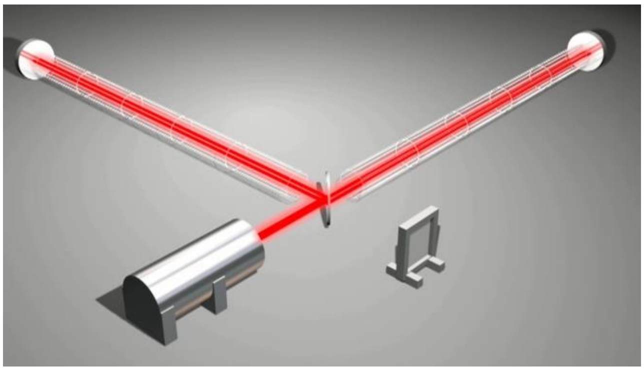

Figure 3.6: LIGO interferometer's shape.

3.4.5 Laser Interferometer Space Antenna (LISA)

This is a planned space-based gravitational wave detector of the National Aeronautics and Space Administration (NASA) together with the European Space Agency (ESA). This mission is planned to be launched in the early 2030*s*. It consists of three spacecrafts separated by 2.5 million of kilometers in a form of an equilateral triangle in space [16]. The working principle is a bit different from the LIGO's and VIRGO's. The laser beam is bounced back and forth between the three spacecrafts and the signals are combined to search for gravitational wave signatures that come from distortions of space-time. LISA will operate in the low frequency range compared to LIGO [1], between 0.1 *mHz* to 1 *Hz*. LISA serves as a space-based observatory that helps in avoiding the noise from earth and due to its long arms, it helps to access regions of the spectrum that are inaccessible from the earth. GW sources that LISA will be able to detect are ultra-compact binaries in our Galaxy, supermassive black hole mergers and extreme mass ratio inspirals[15].

3.4.6 DECi-hertz Interferometer Gravitational-wave Observatory (DECIGO)

This is the planned Japanese space gravitational wave antenna. This is also scheduled to be launched in the 2030*s* after precursor satellite mission *B — DECIGO*. Figure 3.7 shows the DECIGO view [25].

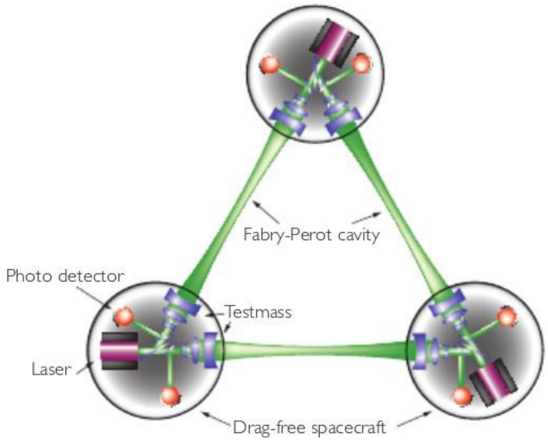

Figure 3.7: Image of DECIGO

DECIGO consists of three drag-free spacecraft arranged in an equilateral triangle separated by 1000 *km* Fabry-Perot cavity (FP) or simply arm lengths. The laser source is supposed to be frequency-doubled Yb:YAG laser[16] with wavelength, λ of 515 nm yifielding an output power of 10 *W* . The mass of the mirror is 100 *kg* with 1 *m* diameter of low-loss high-re ectivity coatings, which enables the nesse of FP cavity to reach 10 with green light [14].

Its aim is to detect gravitational waves from astrophysically and cosmologically significant sources mainly between 0.1 *Hz* and 10 *Hz*. The scope of DECIGO is to bridge the frequency gap between LISA band and other terrestrial detectors band which include LIGO, VIRGO, GEO600, and KAGRA.

[15]Inspirals: Objects in a binary system lost due to emission of GWs.

[16]Yb:YAG: Yttrium aluminum garnet is a laser used for high-power emission.

B — DECIGO is a smaller version of DECIGO. It is expected to be launched late in 2020. This also consists of three spacecraft will have 100 *km* arm lengths orbiting 2000 *km* above the surface of the earth. The mass of the mirrors is 30 *kg* with nesse of 100, the input power is 1 Watt [14].

A measure of how much a gravitational wave distorts space-time is called strain (h), it is equal to the change in displacement between two arms, ΔL, divided by the original distance L. The strain is given by

$$h = \frac{\Delta L}{L} \tag{3.4.1}$$

4. ALTERNATIVE THEORIES OF GRAVITY

Since Einstein proposed the theory of general relativity, other scientists have also proposed oppos- ing theories to compete with it. Among such is the Brans-Dicke theory, the whitehead's theory, the Teleparallelism, the $f(R)$ theories, the Gauss-Bonnet gravity, and the Lovelock theory. All of such are geometric theories of gravity and they use the parameterized post-Newtonian (PPN) formalism. It is a method for solving the equations given by certain theories. In this chapter we will only review the most cited theory which is the Brans-Dicke theory (BD). It is considered as a viable alternative to General Relativity; one which respects Mach's principle.

4.1 Brans-Dicke Theory

In 1960, Carl Brans and Robert Dicke published a theory named after them; the Brans-Dicke (BD) theory. It is a modification or rather generalization of General Relativity (GR). BD theory of gravity theory introduces an additional long-range scalar fifield φ besides the metric tensor $g_{\mu\nu}$ of space-time. The BD theory describes gravitation through a space-time metric $g_{\mu\nu}$ and a massless scalar fifield φ [27].

The fifield equations of the Brans-Dicke theory are given as follows

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{\varphi}T_{\mu\nu} + \frac{\omega}{\varphi^2}(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\lambda\varphi\partial^\lambda\varphi) + \frac{1}{\varphi}(\partial_{\mu\nu}\varphi - g_{\mu\nu}\ \varphi) \quad (4.1.1)$$

$$\varphi = \frac{8\pi}{3+2\omega}T. \quad (4.1.2)$$

The left hand side, $G\mu\nu = R\mu\nu - 1\ Rg\mu\nu$, is the Einstein's tensor. While $R\mu\nu$ is the Ricci tensor, R is the Ricci scalar, and $g\mu\nu$ is the metric tensor. The light speed is c, ω is the dimensionless BD coupling constant, φ is the scalar fifield, $T\mu\nu$ is the energy-momentum tensor associated with the material content is $T = T\mu$, and is the Laplace operator [19]

4.2 Classical Tests of BD Theory

Like GR, BD theory predicts the three classical tests discussed in chapter 3. Light de ection and perihelion shift of Mercury concluded in different results but for the Gravitational Redshift test, both theories are indistinguishable.

4.2.1 Perihelion advance of mercury

Another classical test is the perihelion advance of the mercury's orbit. An exact solution is found in Brans-Dicke, where the precession of a planetary orbit is [20]

$$\delta\varphi_{BD} = \frac{2+2\gamma-\beta}{3}\ \frac{6\pi m^2c^2}{a^2}, \quad (4.2.1)$$

where $\delta\varphi_{BD}$ is the precession for one revolution in BD theory, γ and β are post-Newtonian parameters. Substitution of the value of $p = \frac{a^2}{mc^2}$, gives

$$\delta\varphi_{BD} = \frac{2+2\gamma-\beta}{3}\ \frac{6\pi m}{p},$$

with $m = MG/c^2$ and $p = a_0(1 - e^2)$, then we have

$$\delta\varphi_{BD} = \frac{2+2\gamma-\beta}{3}\ \frac{6\pi MG}{c^2a_0(1-e^2)}. \quad (4.2.2)$$

Equation (4.2.2), shows that the perihelion shift depends upon the post-Newtonian parameters γ and β. In GR theory, $\beta = \gamma = 1$, whereas for the BD theory $\beta = 1$, $\gamma = \omega+1$, therefore equation (4.2.2) becomes

$$\delta\varphi_{BD} = \frac{2\omega+4}{3\omega+6}\ \delta\varphi_{GR},$$

where ω is the Brans-Dicke parameter [1].

4.2.2 Detection angle in the BD theory

For the test of the detection of light in a gravitational fifield, The difference between BD and GR theory for the angle of detection is a factor $\frac{1+\gamma}{2}$. Therefore the detection angle in BD theory is given as

$$\Delta_{BD} = \frac{1+\gamma}{2}\ \frac{4m}{ro},$$

where $\gamma = \frac{\omega+1}{\omega+2}$. Substituting the value of m, we get

$$\Delta_{BD} = \frac{1+\gamma}{2}\ \frac{4MG}{r_0c^2}, \quad (4.2.3)$$

where M is the mass of the sun, G is gravitational constant, r_0 is the nearest approach distance of light ray to the sun, and c

the speed of light. Substitution of value of γ, the detection angle in BD theory becomes

$$\Delta_{BD} = \frac{2\omega + 3}{2\omega + 4} \frac{4MG}{r_0 c^2}. \quad (4.2.4)$$

In short,

$$\Delta_{BD} = \frac{2\omega + 3}{2\omega + 4} \Delta_{GR}. \quad (4.2.5)$$

4.3 The Weak Equivalence Principle (WEP)

The fifirst gravitational experiment that Dicke worked on was the so-called E tv s experiment. E tv s experiment aimed at testing the uniqueness of free fall (the weak equivalence principle) by comparing differences in acceleration of different materials. The principle of uniqueness of free fall states that acceleration due to gravity is the same for all objects regardless of their composition [20]. The experiment tests the equivalence of gravitational mass *mg* and inertial mass *mi*.

In Newton's language, the WEP principle states that any two test bodies must fall with the same acceleration in a given gravitational fifield. Here a test body has the meaning of a neutral body, too small body that its self-gravitational energy can be neglected.

From Newton's second law $F = m_i a$, and gravitation law $F = m_g g$, the acceleration at a given point is given by

$$a = \frac{m_g}{m_i} g \quad (4.3.1)$$

From this relation, acceleration should di er for bodies with different values for the ratio m_g/m_i. E tv s succeeded to show that this ratio between the two masses does not di er from a substance to another more than one part in 10^9. Two objects *A* and *B* of different masses are hanged on the ends of a rod of length *l*, and the rod is suspended by a wire as it is shown in Figure 4.1 [27].

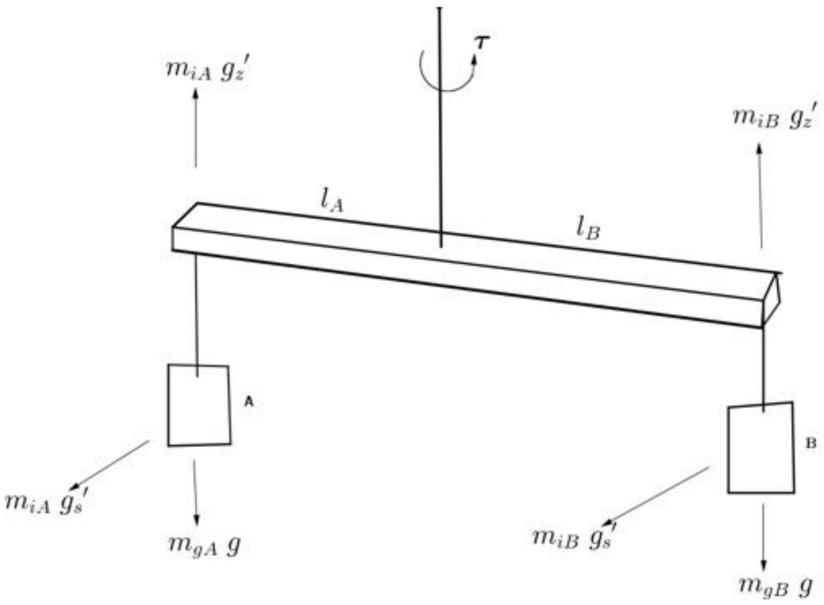

Figure 4.1: Diagram of E tv s experiment.

Here g is the earth's gravitational fifield, g^r_z is the vertical component of the centripetal acceleration due to the earth's rotation, g^r_s is the horizontal component of the centripetal acceleration due to earth's rotation, and l_A and l_B are the effective lever arms for the two weights. At equilibrium the system will bend in such a way that

$$l_A(m_{gA} g - m_{iA} g^r_z) = l_B(m_{gB} g - m_{iB} g^r_z). \quad (4.3.2)$$

The torque τ on the wire to rotate the system.

$$\tau = l_A m_{iA} g^r_s - l_B m_{iB} g^r_s . \tag{4.3.3}$$

From (4.3.2) we can find

$$l_B = l_A \frac{m_{gA} g - m_{iA} g^r_z}{m_{gB} g - m_{iB} g^r_z} = l_A m_{iA} \left(\frac{m_{gA} g}{m_{iA}} - g^r_z \right) \left[m_{iB} \left(\frac{m_{gB} g}{m_{iB}} - g^r_z \right) \right]^{-1} .$$

Equation (4.3.3) becomes

$$\tau = l_A m_{iA} g^r_s - m_{iB} g^r_s \left\{ l_A m_{iA} \left(\frac{m_{gA} g}{m_{iA}} - g^r_z \right) \left[m_{iB} \left(\frac{m_{gB} g}{m_{iB}} - g^r_z \right) \right]^{-1} \right\}$$
$$= l_A m_{iA} g^r_s \left[1 - \left(\frac{m_{gA} g}{m_{iA}} - g^r_z \right) \left(\frac{m_{gB} g}{m_{iB}} - g^r_z \right)^{-1} \right] .$$

Since g^r_z is much less than g torque becomes,

$$\tau = l_A m_{iA} g^r_s \left(1 - \frac{m_{gA}}{m_{iA}} \frac{m_{iB}}{m_{gB}} \right)$$
$$\tau = l_A m_{gA} g^r_s \left(\frac{m_{iA}}{m_{gA}} - \frac{m_{iB}}{m_{gB}} \right) . \tag{4.3.4}$$

E tv s was expecting a twist but he did not detect any (very small). It was because the ratios *mi/mg* in (4.3.4) for the two bodies were equal. They concluded therefore that the earth share the same acceleration to different materials (wood, platinum, copper, etc), which is 5 parts in 10^9. Dicke and his collaborators performed the experiment by improving his method, it become E tv s- Dicke experiment. They tried with sun's gravitational acceleration instead of earth's. It showed that whatever a body is made of does not a ect its acceleration. They concluded that

aluminum and gold fall toward the sun with the same acceleration. The accelerations di ering from each other by at most one part in 10^{11}.

The Brans-Dicke theory obeys the weak equivalence principle, so the experiment supports it as much as it supports GR. However, the strong principle of equivalence does not hold for BD. Because the strong equivalence principle suggests that metric alone determines the effects of gravity and does not have extra fields associated with it. According to it, whilst BD has a scalar fifield φ. The strong principle states that all the laws of physics, as observed locally in a freely falling laboratory, are independent of the location in time or space of the laboratory. The distinction between the strong and weak becomes apparent when clarifying the role of the principle of equivalence in GR and alternative theories of gravity.

5. CONCLUSION

This essay has examined key experimental tests that have confirmed the predictions of General Relativity. Among the earliest validations was the accurate explanation of the anomalous precession of Mercury’s perihelion by Albert Einstein, which provided the fifirst strong empirical support for the theory. This was followed by the landmark 1919 solar eclipse expedition led by Arthur Eddington, which confirmed the bending of starlight by gravity. Further confirmation came from the Pound–Rebka experiment conducted between 1960 and 1965, which precisely measured gravitational redshift using gamma-ray photons in a terrestrial laboratory setting.

The essay also explored modern experimental efforts to detect and analyze gravitational waves, a major prediction of General Relativity. The fifirst direct detection was achieved in 2015 by the LIGO Scientific Collaboration through the GW150914 signal, using the Advanced LIGO detectors located at Hanford and Livingston. Subsequent detections have been enhanced through collaboration with the Virgo Collaboration. Looking ahead, planned space-based observatories such as Laser Interferometer Space Antenna (LISA) and DECIGO aim to detect lower-frequency gravitational waves with greater sensitivity, overcoming many limitations associated with ground-based detectors, particularly environmental noise.

Despite its remarkable success across multiple scales, from solar system observations to cosmological phenomena, General Relativity remains an active area of research and scrutiny. Alternative theories, such as the Brans–Dicke theory, have been proposed to address potential limitations and extend our undefirstanding of gravitation. While these alternatives do not fundamentally contradict General Relativity, they introduce modifications that can be tested through increasingly precise observations.

In conclusion, continuous experimental and observational testing of General Relativity is essential for advancing our undefirstanding of the universe. Future research should expand beyond the Brans–Dicke framework to include a broader range of alternative theories, thereby providing a more comprehensive evaluation of gravitational physics. Such efforts will help determine whether General Relativity remains sufficient in its current form or requires refinement in light of new empirical evidence.

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